IC566: Random Signals and Random Processes (Spring 2019)

Assignment 5 (Due: June 3, 2019)

1. You flip a fair coin 36 times. Let S_{36} denote the number of heads resulting from the 36 tosses. Now you want to calculate $P(S_{36} = 19)$ by using the following two methods.

(a) If you use the binomial coefficient $\binom{n}{k}$ with some integers n and k, what is the exact expression (or number) of the probability?

(b) If you use the central limit theorem and approximate the probability as $P(18.5 < S_{36} < 19.5)$, what is the expression (or number) of the approximate probability? You may use $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$ with some real number z.

2. You want to design a poll for one Yes-or-No question, to which the fraction f of the total population say Yes.

(a) Let X_i be equal to 1 if the ith person polls Yes, and 0 if No. Denote M_n as the sample mean of X_i's with sample size n. Assuming that all X_i's are i.i.d., what are the mean and variance of M_n?

(b) Given that $P(|Z|>1.96) = 0.05$ for a standard normal random variable Z, what is the minimum number of the poll size to guarantee the 95% confidence of less than 1% error?

(c) Now you do not have access to the standard normal distribution table, and should instead use the Chebyshev's inequality

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

What is the minimum number of the poll size to guarantee the 95% confidence of less than 1% error?

3. We wish to estimate the probability of heads, denoted by θ, of a biased coin. We model θ as the value of a random variable Θ that is uniformly distributed at [0, 1]. We consider n independent tosses and let X be the number of heads observed.

(a) Find the PMF of X given Θ, $p_{X|\Theta}(k|\theta)$.

(b) Find the posterior PDF of Θ given X, $p_{\Theta|X}(\theta|k)$.

(Hint: You don't need to calculate $p_X(k)$. Instead, use $p_X(k) = 1/c$ for some c independent of θ.)

(c) Find the MAP estimator of Θ, $\hat{\theta}_{MAP}$.

(d) Find the LMS estimator of Θ, $\hat{\theta}_{LMS}$. You may use the result of $\int_0^1 \theta^k (1-\theta)^{n-k} d\theta = \frac{k!(n-k)!}{(n+1)!}$ in your calculation.

4. Let x be a random sample (i.e., an observed value) of an exponential random variable X with unknown parameter λ.

(a) Assume that λ itself is an exponential random parameter with parameter l. Find the LMS estimator of λ. Use $\int_0^\infty \lambda^n e^{-\lambda(l+x)} d\lambda = \frac{n!}{(l+x)^{n+1}}$.

(b) (optional) Assume that λ is just unknown, but not random. Find the ML estimator of λ