IC666: Discrete Stochastic Processes (Spring 2012)

Solutions to Assignment 1

1. Consider a random experiment of tossing a coin three times.

(a) Find the sample space S_1 if we wish to observe the exact sequences of heads and tails obtained.

(b) Find the sample space S_2 if we wish to observe the number of heads in the three tosses.

Sol)

(a) The sampling space S_1 is given by

$$S_1 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

where, for example, HTH indicates a head on the first and third throws and a tail on the second throw. There are eight sample points in $S_{1.}$

(b) The sampling space S_2 is given by

$$S_2 = \{0, 1, 2, 3\}$$

where, for example, the outcome 2 indicates that two heads were obtained in the three tosses. The

sample space S_2 contains four sample points.

2. Consider the experiment of tossing a fair coin repeatedly and counting the number of tosses required until the first head appears.

(a) Find the sample space of the experiment.

(b) Find the probability that the first head appears on the *k*th toss.

(c) Verify that P(S) = 1.

Sol)

(a) The sample space of this experiment is

$$S_1 = \{ e_1, e_2, e_3, \ldots\}=\{ e_k : k = 1, 2, 3, \ldots \}$$

where e_k is the elementary event that the first head appears on the *k*th toss.

(b) Since a fair coin is tossed, we assume that a head and a tail are equally likely to appear. Then $P(H) = P(T) = \frac{1}{2}$. Let

$$P(e_k)= p_k \qquad k = 1, 2, 3, \ldots$$

Since there are 2^k equally likely ways of tossing a fair coin k times, only one of which consists of (k - 1) tails following a head we observe that

$$P(e_k)= p_k = \frac{1}{2^k} \qquad k = 1, 2, 3, \ldots$$

(c) Using the power series summation formula, we have

$$P(S)= \sum_{k=1}^{\infty} P(e_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2^k}\right)^k = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

3. A lot of 100 semiconductor chips contains 20 that are defective. Two chips are selected at random, without replacement, from the lot.

(a) What is the probability that the first one selected is defective?

(b) What is the probability that the second one selected is defective given that the first one was defective?

(c) What is the probability that both are defective?

Sol)

(a) Let A denote the event that the first one selected is defective. Then by Eq. $P(A)=\frac{n(A)}{n}$ (where n(A) is the number of outcomes belonging to event a and n is the number of sample points in S.)

$$P(A)=\frac{20}{100}=0.2$$

(b) Let B denote the event that the second one selected is defective. After the first one selected is defective, there are 99 chips left in the lot with 19 chips that are defective. Thus, the probability that the second one selected is defective given that the first one was defective is

$$P(B \mid A)=\frac{19}{99}=0.192$$

(c) by $P(A \cap B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$, the probability that both are defective is

$$P(A \cap B) = P(B \mid A)\, P(A) = \left(\frac{19}{99}\right)(0.2) = 0.0384$$

4. Suppose that a laboratory test to detect a certain disease has the following statistics. Let

A = event that the tested person has the disease

B = event that the test result is positive

It is known that

$$P(B \mid A) = 0.99 \quad \text{and} \quad P(B \mid \overline{A}) = 0.005$$

and 0.1 percent of the population actually has the disease. What is the probability that a person has the disease given that the test result is positive?

Sol)

From the given statistics, we have

$$P(A) = 0.001 \quad \text{and} \quad P(\overline{A}) = 0.999$$

The desired probability is P(A | *B*). Thus using Bayes' Rule: $\frac{P(B \mid A)P(A)}{P(B)}$ and Equation: $P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A})$ we obtain

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \overline{A})P(\overline{A})}$$

$$= \frac{(0.99)(0.001)}{(0.99)(0.001) + (0.005)(0.999)} = 0.165$$

Note that in only 16.5 percent of the cases where the tests are positive will the person actually have the disease even though the test is 99 percent effective in detecting the disease when it is, in fact, present.

5. Consider the binary communication channel shown in Fig 1.The channel input symbol X may assume the state 0 or the state 1, and, similarly, the channel output symbol Y may assume either the state 0 or the state 1. Because of the channel noise, an input 0 may convert to an output 1 and vice versa. The channel is characterized by the channel transition probabilities p_0, q_0, p_1, and q_1, defined by

$$p_0 = P(y_1 \mid x_0) \qquad \text{and} \qquad p_1 = P(y_0 \mid x_1)$$

$$q_0 = P(y_0 \mid x_0) \qquad \text{and} \qquad q_1 = P(y_1 \mid x_1)$$

where x_0 and x_1 denote the events ($X = 0$) and ($X = 1$), respectively, and y_0 and y_1 denote the events ($Y = 0$) and ($Y = 1$), respectively. Note that $p_0 + q_0 = 1 = p_1 + q_1$. Let $P(x_0) = 0.5$, $p_0 = 0.1$, and $p_1 = 0.2$.

(a) Find $P(y_0)$ and $P(y_1)$.

(b) If a 0 was observed at the output, what is the probability that a 0 was the input state?

(c) If a 1 was observed at the output, what is the probability that a 1 was the input state?

(d) Calculate the probability of error P_e.

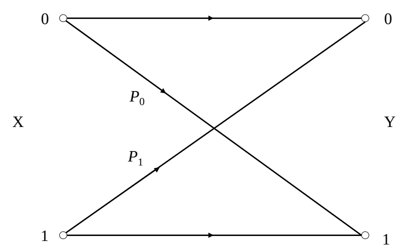

Fig. 1

Sol)

(a) We note that

$$P(x_1)=1-P(x_0)=1-0.5=0.5$$

$$P(y_0 \mid x_0) = q_0 = 1 - p_0 = 1 - 0.1 = 0.9$$

$$P(y_1 \mid x_1) = q_1 = 1 - p_1 = 1 - 0.2 = 0.8$$

Using Equation: $P(B) = \sum_{i=1}^{n} P(B \cap A_i) P(A) = \sum_{i=1}^{n} P(B \mid A_i) P(A_i)$ we obtain

$$P(y_0) = P(y_0 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.9(0.5) + 0.2(0.5) = 0.55$$

$$P(y_1) = P(y_1 \mid x_0)P(x_0) + P(y_1 \mid x_1)P(x_1) = 0.1(0.5) + 0.8(0.5) = 0.45$$

(b) Using Bayes' rule, we have

$$P(x_0 \mid y_0) = \frac{P(x_0)P(y_0 \mid x_0)}{P(y_0)} = \frac{(0.5)(0.9)}{0.55} = 0.818$$

(c) Similarly,

$$P(x_1 \mid y_1) = \frac{P(x_1)P(y_1 \mid x_1)}{P(y_1)} = \frac{(0.5)(0.8)}{0.45} = 0.889$$

(d)The probability of error is

$$P_e = P(y_1 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.1(0.5) + 0.2(0.5) = 0.15$$

6. Let A and B be events in a sample space S. Show that if A and B are independent, then so are (a) A and $\overline{B}$, (b) $\overline{A}$ and B, and (c) $\overline{A}$ and $\overline{B}$.

Sol)

(a) we have

$$P(A) = P(A \cap B) + P(A \cap \overline{B})$$

Since A and B are independent, we obtain

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = P(A) - P(A)P(B)$$
$$= P(A)[1 - P(B)] = P(A)P(\overline{B}) \qquad (1)$$

Thus, by definition: $P(A \cap B) = P(A)P(B)$ if A and B are independent, A and $\overline{B}$ are independent.

(b) Interchanging A and B in Equation (1)

$$P(B \cap \overline{A}) = P(B)P(\overline{A})$$

which indicates that $\overline{A}$ and B are independent.

(C) we have

$$\begin{aligned} P(\overline{A} \cap \overline{B}) &= P[(\overline{A \cup B})] \\ &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= 1 - P(A) - P(B)[1 - P(A)] \\ &= [1 - P(A)][1 - P(B)] \\ &= P(\overline{A})(\overline{B}) \end{aligned}$$

Hence, $\overline{A}$ and $\overline{B}$ are independent.