IC666: Discrete Stochastic Processes (Spring 2012)

Solutions to Assignment 2

1. Using a biased coin to make an unbiased decision.

Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased (though the bias is not known exactly). How can they use the biased coin to make a decision so that either option (opera or the movies) is equally likely to be chosen?

Sol)

Flip the coin twice. If the outcome is heads-tails, choose the opera.

if the outcome is tails-heads, choose the movies. Otherwise, repeat the process, until a decision can be made. Let A_k be the event that a decision was made at the kth round. Conditional on the event A_k, the two choices are equally likely, and we have

$$\mathrm{P(opera)} = \sum_{k=1}^{\infty} \mathrm{P}(opera \mid A_k)\mathrm{P}(A_k) = \sum_{k=1}^{\infty} \frac{1}{2}\mathrm{P}(A_k) = \frac{1}{2}$$

We have used here the property $\sum_{k=1}^{\infty} \mathrm{P}(A_k) = 1$, which is true as long as P(heads) > 0 and P(tails) > 0.

2. St.Petersburg paradox. You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is *n*, you receive 2^n dollars. What is the expected amount that you will receive? How much would you be willing to pay to play this game?

Sol)

The expected value of the gain for a single game is infinite since if X is your gain, then

$$E[X] = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \sum_{k=1}^{\infty} 1 = \infty$$

Thus if you are faced with the choice of playing for given fee f or not playing at all, and your objective is to make the choice that maximizes your expected net gain, you would be willing to pay any value of f. However, this is in strong disagreement with the behavior of individuals. In fact experiments have shown that most people are willing to pay only about \$20 to \$30 to play the game. The discrepancy is due to a presumption that the amount one is willing to pay is determined by the expected gain. However, expected gain does not take into account a person's attitude towards risk taking.

3. Alice passes through four traffic lights on her way to work, and each light is equally likely to be green or red, independently of the others.

(a) What is the PMF, the mean, and the variance of the number of red lights that Alice encounters?

(b) Suppose that each red light delays Alice by exactly two minutes. What is the variance of Alice's commuting time?

Sol)

(a) Let X be the number of red lights that Alice encounters. The PMF of X is binomial with $n = 4$ and $p = 1/2$.

The mean and the variance of X are $E[X] = np = 2$ and $\text{var}(X) = np(1 - p) = 4 \cdot (1/2) \cdot (1/2) = 1$.

(b) The variance of Alice's commuting time is the same as the variance of the time by which Alice is delayed by the red lights. This is equal to the variance of $2X$, which is $4\text{var}(X) = 4$.

4. Let X and Y be independent random variables that take values in the set {1,2,3}. Let V =2X + 2Y and $W = X - Y$.

(a) Assume that P({$X = k$}) and P({$Y = k$}) are positive for any $k \in \{1,2,3\}$. Can V and W be independent? Explain. (No calculations needed.)

For the remaining parts of this problem, assume that X and Y are uniformly distributed on {1, 2, 3}

(b) Find and plot $p_V(v)$. Also, determine E[V] and var(V).

(c) Find and show in a diagram $p_{V,W}(v, w)$.

(d) Find E[V | W > 0].

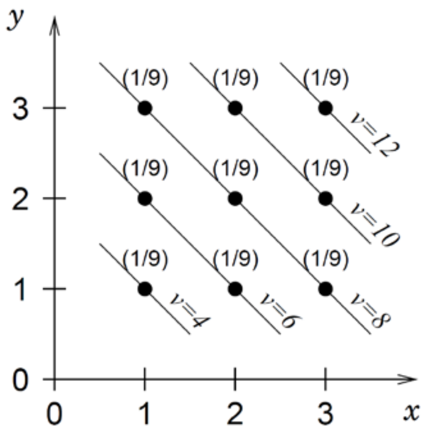
(e) Find the conditional variance of W given the event {V =8}.

(f) Find and plot the conditional PMF $p_{X|V}(x|v)$, for all values.

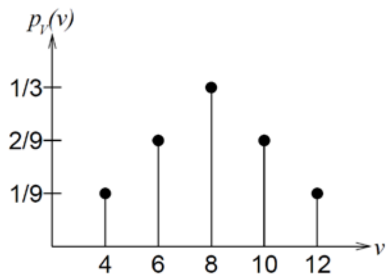
Sol)

(a) V and W cannot be independent. Knowledge of one random variable gives information about the other. For instance, if V = 12 we know that W = 0.

(b) We begin by drawing the joint PMF of X and Y .



X and Y are uniformly distributed so each of the nine grid points has probability 1/9. The lines on the graph represent areas of the sample space in which V is constant. This constant value of V is indicated on each line. The PMF of V is calculated by adding the probability associated with each grid point on the appropriate line.



By symmetry (or direct calculation), $\boxed{E[V] = 8}$. The variance is:

$$var(V) = (4 - 8)^2 \cdot \frac{1}{9} + (6 - 8)^2 \cdot \frac{2}{9} + 0 + (10 - 8)^2 \cdot \frac{2}{9} + (12 - 8)^2 \cdot \frac{1}{9} = \frac{16}{3}$$

Alternatively, note that V is twice the sum of two independent random variables, $V = 2(X + Y)$, and hence

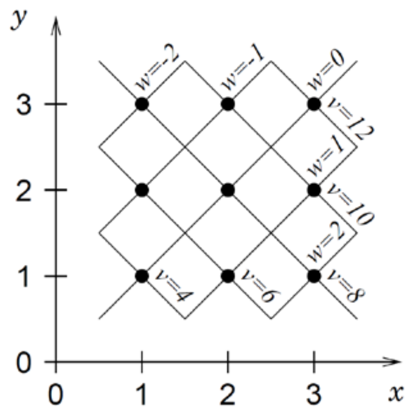
$$var(V) = var(2(X + Y)) = 2^2 var(X + Y) = 4(var(X) + var(Y)) = 4 \cdot 2var(X) = 8var(X)$$

(Note the use of independence in the third equality; in the fourth one we use the fact that X and Y are identically distributed, therefore they have the same variance). Now, by the distribution of X, we can easily calculate that

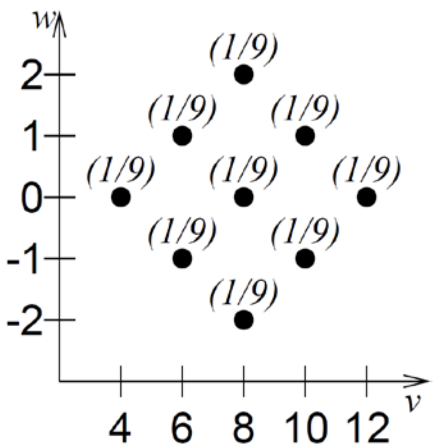
$$var(X) = \frac{1}{3}(1 - 2)^2 + \frac{1}{3}(2 - 2)^2 + \frac{1}{3}(3 - 2)^2 = \frac{2}{3}$$

so that in total $var(V) = \frac{16}{3}$, as before.

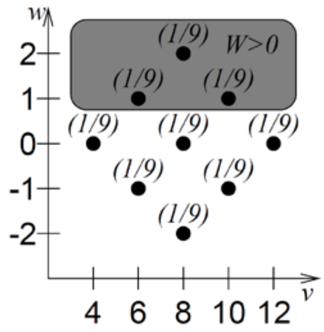
(c) We start by adding lines corresponding to constant values of W to our first graph in part (b):



Again, each grid point has probability 1/9. Using the above graph, we get $p_{V,W}(v, w)$

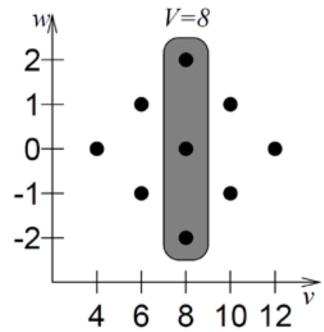


(d) The event W > 0 is shaded below:



By symmetry (or an easy calculation), $\boxed{E[V \mid W > 0] = 8}$.

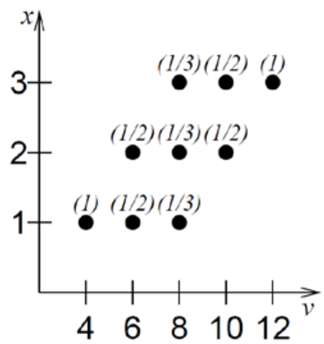
(e) The event $\{V = 8\}$ is shaded below:



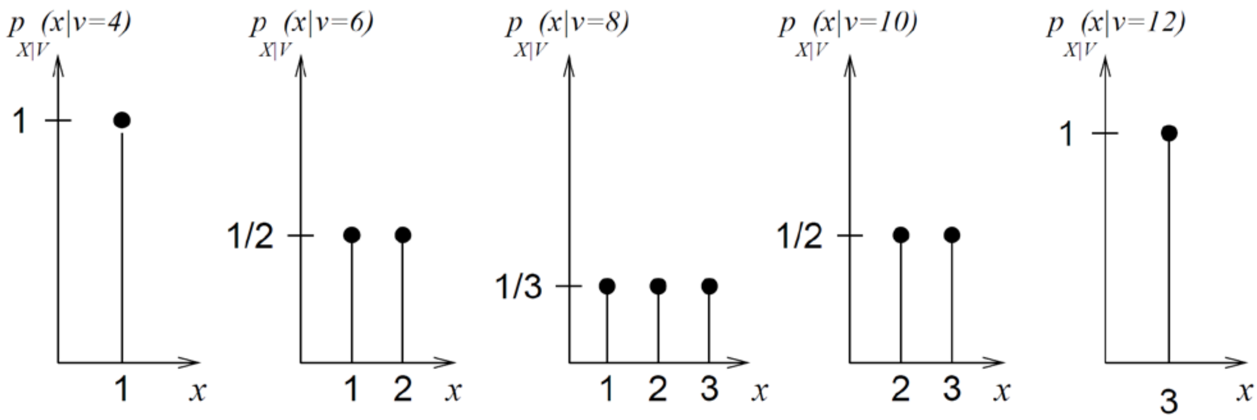
When V = 8, W can take on values in the set $\{-2, 0, 2\}$ with equal probability. By symmetry (or an easy calculation), $E[W \mid V = 8] = 0$. The variance is:

$$var(W \mid V = 8) = (-2 - 0)^2 \cdot \frac{1}{3} + (0 - 0)^2 \cdot \frac{1}{3} + (2 - 0)^2 \cdot \frac{1}{3} = \frac{8}{3}$$

(f) Please refer to the first graph in part (b). When V = 4, X = 1 with probability 1. When V = 6, X can take on values in the set $\{1, 2\}$ with equal probability. Continuing this reasoning for the other values of V , we get the following conditional PMFs, which we plot on one set of axes.



Note that each column of the graph is a separate conditional PMF and that the probability of each column sums to 1. This part of the problem illustrates an important point. $p_{X|V}(x \mid v)$ is actually not a single PMF but a family of PMFs.



$p_{X|V}(x|v=4)$
1
1
x
$p_{X|V}(x|v=6)$
1/2
1
2
x
$p_{X|V}(x|v=8)$
1/3
1
2
3
x
$p_{X|V}(x|v=10)$
1/2
2
3
x
$p_{X|V}(x|v=12)$
1
3
x

5. An information source generates symbols at random from. a four-letter alphabet (a, b, c, d} with Probabilities $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{4}$, and $P(c) = P(d) = \frac{1}{8}$. A coding scheme encodes these symbols Into binary codes as follows:

a	0
b	10
c	110
d	111

Let X be the r.v. denoting the length of the code, that is, the number of binary symbols (bits).

(a) What is the range of X?

(b) Assuming that the generations of symbols are independent, find the probabilities P(X = I), P(X = 2), P(X = 3), and P(X > 3).

Sol)

(a) The range of X is $R_x = \{1,2, 3)$

(b) $P(X = 1) = P[\{a\}] = P(a) = \frac{1}{2}$

$P(X = 2) = P[\{b\}] = P(b) = \frac{1}{4}$

$P(X = 3) = P[\{c, d\}] = P(c) + P(d) = \frac{1}{4}$

$P(X > 3) = P(\phi) = 0$

6. Suppose a discrete r.v. X has the following pmfs:

$$Px(1) = \frac{1}{2} \qquad Px(2) = \frac{1}{4} \qquad Px(3) = \frac{1}{8} \qquad Px(4) = \frac{1}{8}$$

(a) Find and sketch the cdf $F_X(x)$ of the r.v. X.

(b) Find (i) $P(X \le 1)$, (ii) $P(1 < X \le 3)$, (iii) $P(1 \le X \le 3)$.

Sol)

Equation cdf $F_X(x)$ of the r.v. X.

$$F_x(x) = P(X \le x) = \sum_{x_k \le x} p_x(x_k)$$

We obtain

$$F_x(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$

Which is sketched in Fig 1.

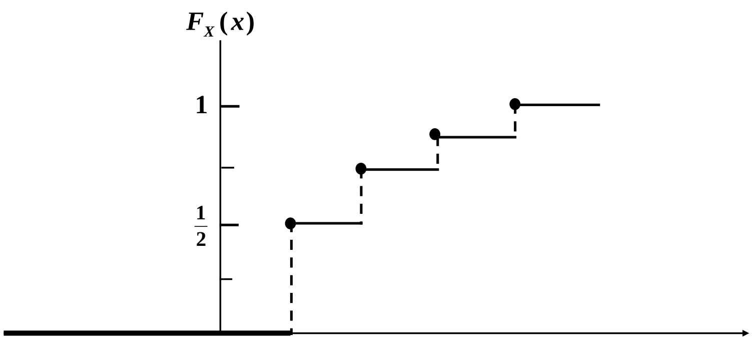


Figure 1

(b)

(i) P(X ≤ 1)

$$P(X < 1) = F_X(1^-) = 0$$

(ii) P(l < X ≤ 3)

$$P(1 < X \leq 3) = F_X(3) - F_X(1) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

(iii) P(l ≤ X ≤ 3)

$$P(1 \leq X \leq 3) = P(X = 1) + F_X(3) - F_X(1) = \frac{1}{2} + \frac{7}{8} - \frac{1}{2} = \frac{7}{8}$$